ON THE CONDITION OF REDUCIBILITY OF ANY GROUP OF LINEAR SUBSTITUTIONS

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LET

$$\left.\begin{aligned} x_i' &= \sum_1^n \alpha_{ij} x_j \\ x_i' &= \sum_1^n \beta_{ij} x_j \\ x_i' &= \sum_1^n \gamma_{ij} x_j \end{aligned}\right\} \quad (i = 1, 2, \ldots, n)$$

be the substitutions A, B, C of a group G of linear substitutions, Then, if

$$AB = C,$$

where, in the product AB, the symbols are to be read from left to right,

$$\gamma_{ij} = \sum_k \beta_{ik} \alpha_{kj}. \tag{i}$$

If in these equations the α's are regarded as the original variables and the γ's as the transformed variables, then they define a group of linear substitutions in n^2 variables simply isomorphic with G, the substitution given being that corresponding to B. Moreover this group is reducible, transforming each set of n variables

$$\alpha_{kj} \quad (k = 1, 2, \ldots, n)$$

among themselves exactly as G transforms the x's.

If G is reducible, it is possible to choose new variables

$$\xi_i = \Sigma A_{ij} x_j \quad (i = 1, 2, \ldots, n)$$

so that the first r $(< n)$ of them are transformed among themselves by every substitution of the group. The transformed group so set up may be represented by

$$\xi_i' = \sum_1^n \alpha_{ij}' \xi_j \quad (i = 1, 2, \ldots, n),$$

the substitution given corresponding to A; and in this form

$$\alpha_{ij}' = 0 \quad (i = 1, 2, \ldots, r;\ j = r+1, r+2, \ldots, n)$$

for every substitution of the group. But in this transformed form of the group

$$\alpha'_{ij} = \sum_{st} B_{st}\,\alpha_{st},$$

where the B's are constants depending on the equations defining the new variables.

Hence, if G is reducible, so that a set of r new variables may be chosen which are transformed only among themselves, then the coefficients of every substitution of G must satisfy a system of $r(n-r)$ independent linear homogeneous equations.

Conversely, suppose that the coefficients of G satisfy a system of s independent linear equations

$$\Sigma C_{ij}^{(k)}\beta_{ij} = 0 \quad (k = 1, 2, \ldots, s). \tag{ii}$$

If in the equations (i) the n^2 variables are replaced by n^2 independent linear functions of themselves, of which the first s are

$$A_k = \Sigma C_{ij}^{(k)}\alpha_{ij} \quad (k = 1, 2, \ldots, s),$$

the new form of the group defined by them will be

$$A'_i = \sum_{k=1}^{k=n^2} B_{ik}A_k \quad (i = 1, 2, \ldots, n^2),$$

where the B's are linear functions of the β's with constant coefficients depending on the C's. If in any given set of these equations, specifying a particular substitution of the group, the α's which occur in the definition of the A's are replaced by the coefficients of an arbitrarily chosen substitution of G, the equations become identities. Now, when this is done,

$$A_1 = A_2 = \ldots = A_s = A'_1 = \ldots = A'_s \equiv 0.$$

Hence the first s equations become

$$\sum_{k=s+1}^{k=n^2} B_{ik}A_k = 0 \quad (i = 1, 2, \ldots, s). \tag{iii}$$

Now, if for any given substitution of the group on the n^2 variables the B's of these equations are not identically zero, these s equations are equivalent to a further set of linear homogeneous equations connecting the coefficients of every substitution of G.

Hence, if the set of s equations (ii) are the only linear independent equations connecting the coefficients of every substitution of G, the B's in the equations (iii) must be zero for every substitution of the group on the n^2 variables, and the s symbols

$$A_1, A_2, \ldots, A_s$$

are transformed among themselves by every substitution of this group.

Now it has been seen above that each set of n a's which have the same second suffix are transformed among themselves by every substitution of the group. Hence for each j the set of s linear functions

$$\sum_{i=1}^{i=n} C_{ij}^{(k)} a_{ij} \quad (k = 1, 2, \ldots, s)$$

are transformed among themselves by every substitution of the group. If for any second suffix j which actually occurs this set of s functions is linearly equivalent to r $(< n)$, then a set of r linear functions of

$$a_{1j},\ a_{2j},\ \ldots,\ a_{nj}$$

exists which are transformed among themselves; and therefore there is a corresponding set of r linear functions of the x's which are transformed among themselves by the substitutions of G, *i.e.*, G is reducible.

If for each second suffix that occurs the s functions were equivalent to n (in which case s must be a multiple of n), successive sets of symbols with the same second suffix might be eliminated from equations (ii) till there remain only n equations. These may be brought to the form

$$\left.\begin{array}{l} \beta_{1j_1}+B_{1j_2}+\ldots = 0 \\ \beta_{2j_1}+B_{2j_2}+\ldots = 0 \\ \ldots \quad \ldots \quad \ldots \\ \beta_{nj_1}+B_{nj_2}+\ldots = 0 \end{array}\right\}, \qquad \text{(iv)}$$

where $B_{1j_2}, B_{2j_2}, \ldots, B_{nj_2}$ are n linearly independent functions of

$$\beta_{1j_2},\ \beta_{2j_2},\ \ldots,\ \beta_{nj_2}.$$

Now, in the group on the n^2 variables, symbols with the same second suffix are transformed among themselves. Hence, if A_{rj_2} is the same function of $a_{1j_2}, a_{2j_2}, \ldots, a_{nj_2}$ that B_{rj_2} is of the corresponding β's, then

$$\begin{array}{l} a_{1j_1}+A_{1j_2}+\ldots, \\ a_{2j_1}+A_{2j_2}+\ldots, \\ \ldots \quad \ldots \quad \ldots \\ a_{nj_1}+A_{nj_2}+\ldots \end{array}$$

are transformed among themselves, and therefore $a_{1j_1}, a_{2j_1}, \ldots, a_{nj_1}$ and $A_{1j_2}, A_{2j_2}, \ldots, A_{nj_2}$ undergo the same transformation for every substitution of the group.

Hence also $\qquad a_{1j_2},\ a_{2j_2},\ \ldots,\ a_{nj_2}$

and $\qquad A_{1j_2},\ A_{2j_2},\ \ldots,\ A_{nj_2}$

undergo the same transformation for every substitution of the group. This is the same as the statement that every substitution on

$$a_{1j_2},\ a_{2j_2},\ \ldots,\ a_{nj_2}$$

is permutable with

$$a'_{1j_2} = A_{1j_2},\quad a'_{2j_2} = A_{2j_2},\quad \ldots,\quad a'_{nj_2} = A_{nj_2},$$

and therefore either that

$$A_{rj_2} = ka_{rj_2}\quad (r = 1, 2, \ldots, n),$$

where k is a constant, or that the group on

$$a_{1j_2},\ a_{2j_2},\ \ldots,\ a_{nj_2}$$

is reducible.

Now, if for each set of symbols, such as B_{rj_2}, which occur in the equations (iv), $B_{rj_2} = k\beta_{rj_2}$, the equations would be

$$\begin{aligned} &\beta_{1j_1}+k\beta_{1j_2}+l\beta_{ij_3}+\ldots = 0,\\ &\beta_{2j_1}+k\beta_{2j_2}+l\beta_{2j_3}+\ldots = 0,\\ &\ldots \qquad \ldots \qquad \ldots \qquad \ldots \end{aligned}$$

implying that the determinant of the substitution is zero. This is not the case. Hence the group in a set of symbols with the same second suffix, and therefore also the group on the x's, is reducible.

The result may be stated as follows :—

Theorem.—The necessary and sufficient condition that a group of linear substitutions on a finite number of symbols should be reducible is that one or more homogeneous linear equations should be satisfied by the coefficients of every substitution of the group.

It follows that, if a group of linear substitutions in n variables has a set of n^2 substitutions $A_1, A_2, \ldots, A_{n^2}$, such that

$$\begin{vmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \ldots & a_{nn}^{(1} \\ a_{11}^{(2)} & a_{12}^{(2)} & \ldots & a_{nn}^{(2} \\ \ldots & \ldots & \ldots & \\ a_{11}^{(n^2)} & a_{12}^{(n^2)} & \ldots & a_{nn}^{(n^2)} \end{vmatrix} \neq 0,$$

the group is irreducible ; and, conversely, if the group is irreducible, there must be such a set of n^2 substitutions.

If, in analogy with groups of finite order, the sum of the coefficients in the leading diagonal of a substitution A, viz.,

$$a_{11}+a_{22}+\ldots+a_{nn},$$

is called the characteristic of the substitution and is denoted by χ_A, the preceding condition can be expressed in terms of the characteristics of the products, two and two, of the set of n^2 substitutions. In fact, apart from sign, the square of the preceding determinant is the determinant for which the element in the k-th row and l-th column is

$$\sum_{ij} a_{ij}^{(k)} a_{ji}^{(l)}.$$

But this is the sum of the coefficients in the leading diagonal of $A_k A_l$.

The necessary and sufficient condition, then, that a group of linear substitutions in n variables should be irreducible is that it should be possible to choose from it a set of n^2 substitutions for which the determinant

$$| \chi_{A_k A} |$$

does not vanish.